Lecture 2
Thursday, October 3, 2019 3:40 PM

We start in Ch II. (Ch. I is left for self study.) Much of this will be familiar to many students.

1. Metrics (distances) and metric spaces.

Def. • A metric space (X, d) is a set X equipped w/ a metric (distance) d.
• The metric d satisfies
(i) $d(x,y) \geq 0$
(ii) $d(x,y) = 0 \iff x = y$
(iii) $d(x,y) = d(y,x)$
(iv) Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

Ex ① $X = \mathbb{C}$, $d(z,w) = |z-w|$ (Standard Euclidean metric)
② $X = \mathbb{C}_\infty$, d - Fubini-Study
$d(z,w) = \dfrac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$

2. Subspaces.
If $A \subseteq X$, then (A, d) is a metric space w/ the inherited metric.
(This is obvious.)

3. Open sets.
For $x \in X$, $r > 0$, the open ball centered at x of radius r is
$$B(x,r) := \{y \in X : d(x,y) < r\}.$$
Note: If $A \subseteq X$, then the open balls in (A,d) are $B(x,r) \cap A$.

Def ① A set $G \subseteq X$ is open if $\forall x \in G$ $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq G$.
② A set $F \subseteq X$ is closed if $X \setminus F$ is open.

Note: If $G \subseteq X$ is open, $A \subseteq X$, then $G \cap A$ is open in (A,d). Same for closed sets.

Basic Props of open sets: Let (X,d) be metric space.
(i) X and $\emptyset$ are open
(ii) If $G_1, \dots, G_n$ are open $\Rightarrow \bigcap_{k=1}^{n} G_k$ is open (finite intersections)
(iii) If $\{G_\alpha\}_{\alpha \in I}$ are open $\Rightarrow \bigcup_{\alpha \in I} G_\alpha$ is open (arbitrary unions).

Proof Exercise or see Conway.

... sets are complements of

(iii) If $\{G_\alpha\}_{\alpha \in I}$ ... $\alpha \in I$

<u>Proof</u>. Exercise or see Conway.

By using basic set theory and the fact that closed sets are complements of open sets, we have

<u>Basic Props of closed sets</u>:

(i) X and $\emptyset$ are closed.

(ii) If $\{F_\alpha\}_{\alpha \in I}$ are closed $\Rightarrow \bigcap_{\alpha \in I} F_\alpha$ is closed.

(iii) If $F_1, \dots, F_n$ are closed $\Rightarrow \bigcup_{k=1}^{n} F_k$ is closed.

<u>Three operations</u>.

① If $A \subseteq X$, then $\operatorname{int} A$ is the open set
$$\operatorname{int} A = \bigcup \{G \subseteq A \text{ open}\} = \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \subseteq A\}$$

② If $A \subseteq X$, then $\overline{A}$ is the closed set
$$\overline{A} = \bigcap \{F \text{ closed} : A \subseteq F\}$$

③ If $A \subseteq X$, then ∂A is the closed set
$$\partial A = \overline{A} \cap \overline{(X - A)} = \overline{A} \setminus \operatorname{int} A.$$

Numerous relations between these can be deduced by basic set Theory (see Conway for a list).

<u>Def</u>. A set $A \subseteq X$ is <u>dense</u> if $\overline{A} = X$.

4. <u>Connectedness</u>

<u>Def</u>. ① A metric space (X, d) is <u>connected</u> if $X, \emptyset$ are the only subsets that are both open and closed.

② $A \subseteq X$ is <u>connected</u> if (A, d) is connected as a metric space.

<u>Ex</u>. Let $A, B \subseteq \mathbb{C}$ be two disjoint $\underbrace{\text{open}}^{\text{nonempty}}$ sets $(A \cap B = \emptyset)$. Then $A \cup B$ is not connected. Clearly, both A, B are open in $A \cup B$ since $(A \cap B) \cap A = A$ and similarly for B. But ... $A = (A \cup B) \setminus B$. Thus,

Then $A \cup B$ is not conn... ... and similarly for B. But open in $A \cup B$ since $(A \cap B) \cap A = A$ and similarly for B. But
than A, B are also closed since e.g. $A = (A \cup B) \setminus B$. Thus,
A (and B for that matter) is open and closed, nonempty and not equal to $A \cup B$. Thus, $A \cup B$ is not connected.

Rem. Another way to characterize connectness is to say (X, d) is not connected if $X = A \cup B$, where A, B are open and $A \cap B = \emptyset$.
Note that since A, B are both open, $A \cap B = \emptyset \Rightarrow$ both $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. (Ex. Prove this!)

→ Lecture 2 ended here. Lecture 3 will begin w/ the below.

Prop 1. A subset $A \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ A is an interval.

Proof. See Conway.

Thm 1. An open set $G \subseteq \mathbb{C}$ is connected $\Leftrightarrow$ $\forall a, b \in G$ $\exists$ polygonal path $P = \bigcup_{k=1}^{n} [z_{k-1}, z_k]$ in G connecting a to b, i.e. $z_0 = a$, $z_n = b$.

Rem. Can replace polygonal path w/ polygonal path using only horizontal and vertical line segments.

Proof. $\Leftarrow$. Suppose G has polygonal path property but is not connected. Then $G = A \cup B$, $A \cap B = \emptyset$ and A, B both open and nonempty. Pick $a \in A$, $b \in B$ and connect them by a polygonal path P.
An inductive argument show that $\exists$ at least on line segment with one endpoint in A and one in B. Thus WLOG, $P = [a, b]$.
Parametrize $[a, b]$ by $\gamma(t) = (1-t)a + tb$, $t \in [0, 1]$.
Let $T_A := \{t : \gamma(t) \in A\}$, $T_B = \{t : \gamma(t) \in B\}$.
T_A, T_B are nonempty and not equal to $[0,1]$ since

Let $T_A := \{t : \gamma(t) \in A\}$, T_B ...

Both are nonempty and not equal to $[0,1]$ since $a \in A$, $b \in B$, and $T_A \cup T_B = [0,1]$.

Claim: Both T_A, T_B are open.

Why? By symmetry, suffices to show T_A is open.
Let $s \in T_A$. Since $A \cap B = \emptyset$, $\exists\, \varepsilon > 0$ s.t. $B(\gamma(s), \varepsilon) \cap B = \emptyset$. But $\exists\, \delta > 0$ s.t. $\gamma(t) \in B(\gamma(s), \varepsilon)$ if $|s - t| < \delta$ (Continuity) $\Rightarrow \gamma(t) \in T_A$. $\Rightarrow$ T_A open.

Now, the claim contradict that $[0,1]$ is connected (according to Prop 1); thus, G is connected.

$\Rightarrow$: G open, connected. Pick $a \in G$ arbitrary.
Suffices to show that every $b \in G$ can be connected to a by a polygonal path.
Set $B = \{b \in G : b \text{ can be connected}\}$.
Suffices to show B is open, closed, and nonempty. ← trivial

- B open: Let $b \in B$. Since G is open $\exists\, \varepsilon > 0$ s.t. $B(b, \varepsilon) \subseteq G$. But then if $b' \in B(b, \varepsilon)$, $[b, b'] \subseteq G$, so by adding this segment to path connecting to b, we conclude $b' \in B \Rightarrow B(b, \varepsilon) \subseteq B \Rightarrow B$ open.

- B closed: Let $c \in G \setminus B$. There is $B(c, \varepsilon) \subseteq G$. But $B(c, \varepsilon) \subseteq G \setminus B$, b/c if $\exists\, b \in B \cap B(c, \varepsilon)$, then $[b, c] \subseteq B(c, \varepsilon) \underset{\text{as above}}{\Rightarrow} c \in B$. Thus,

$G \setminus B$ is open, so B is closed.

This proves $B = G$.

$\square$